

Approximate amenability of Segal algebras II

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Abstract

We prove that every proper Segal algebra of a SIN group is not approximately amenable.

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Gourdeau in [5] showed that a Banach algebra A is [amenable](#) if and only if every bounded derivation $D : A \rightarrow X$ for any Banach A -bimodule X can be approximated by a net of inner derivations. A weaker version of this notion is [approximate amenability](#) of Banach algebras that is, a Banach algebra A is approximately amenable if and only if every bounded derivation $D : A \rightarrow X^*$ for every Banach dual A -bimodule X^* can be approximated by a net of inner derivations. The concept of approximate amenability first was introduced and studied in [4].

Similar to amenability, different algebras were studied for their approximate amenability property including [Segal algebras](#) (for the definition of Segal algebras and their basic properties, look at [9].) In [3], Dales and Loy studied some specific Segal algebras on the commutative groups \mathbb{T} and \mathbb{R} . They proved that those Segal algebras are not approximately amenable; consequently, they suggested that the same should be true for every [proper](#) Segal algebra on these groups. We call a Segal algebra proper if it not equal to the group algebra.

Subsequently, Choi and Ghahramani, [2], proved that this conjecture is true by showing that proper Segal algebras on \mathbb{T}^d and \mathbb{R}^d are not approximately amenable (for any dimension d). To do so, they developed a criterion for “ruling out approximate amenability” of Banach algebras. Ghahramani, in the Banach algebra 2011 conference, conjectured that every proper Segal algebra of a locally compact group cannot be approximately amenable.

In [1], the author applied the criterion developed in [2] to show that in fact, every proper Segal algebra of a locally compact abelian group is not approximately amenable. Also, applying the hypergroup structure on the dual of compact groups, it was proved that for some classes of compact groups, including $SU(2)$, every proper Segal algebra is not approximately amenable.

In this short manuscript we prove that this conjecture is actually true for every SIN group. Recall that a locally compact group G is called a [SIN group](#) if there exists a topological basis of

conjugate invariant neighbourhoods of the identity element of the group G . This class of locally compact groups includes abelian, compact, and discrete groups.

Theorem 1 *Every proper Segal algebra of a SIN group is not approximately amenable.*

We prove this theorem, for a generalized version of Segal algebras, called **abstract Segal algebras**. A Banach algebra $(B, \|\cdot\|_B)$ is an abstract Segal algebra of a Banach algebra $(A, \|\cdot\|_A)$ if B is a dense left ideal in A , there exists $C > 0$ such that $\|b\|_A \leq C\|b\|_B$ (for each $b \in B$), and there exists $M > 0$ such that $\|ab\|_B \leq M\|a\|_A\|b\|_B$ for all $a, b \in B$. We call B a **proper** abstract Segal algebra of A if $B \neq A$. It is clear that every (proper) Segal algebra of a locally compact group G is a (proper) abstract Segal algebra of $L^1(G)$.

Let A be a commutative Banach algebra and let $\Delta(A)$ denote the **Gelfand spectrum** of A and for each $a \in A$, \widehat{a} is the Gelfand transform of a . For definitions related to the Gelfand spectrum of commutative Banach algebras, we refer to [6]. We denote the set of all elements $a \in A$ such that $\text{supp}(\widehat{a})$ is compact by A_c . A semisimple commutative Banach algebra A is called a **Tauberian algebra** when A_c is dense in A .

For a Banach algebra A and a constant $D > 0$, A has a **D -bounded approximate identity** if there is a net $(e_\alpha)_\alpha \subseteq A$ such that for every $a \in A$, $\|ae_\alpha - a\|_A \rightarrow 0$, $\|e_\alpha a - a\|_A \rightarrow 0$, and $\sup_\alpha \|e_\alpha\|_A \leq D$. Note that if A is a unital commutative Banach algebra with the unit $e \in A$, then u is constantly one on $\Delta(A)$. The following lemma shows that the existence of a bounded approximate identity approximately plays a similar role for a regular Tauberian algebra.

Lemma 2 *Let A be a regular commutative Tauberian Banach algebra. Then A has a D -bounded approximate identity if and only if for each compact set $K \subseteq \Delta(A)$ and $\epsilon > 0$, there is some $a_{K,\epsilon} \in A$ such that $\|a_{K,\epsilon}\|_A \leq D$ and $\widehat{a}_{K,\epsilon}|_K \equiv 1$.*

Proof. Suppose that $(e_\alpha)_\alpha$ is a bounded approximate identity of A such that $\|e_\alpha\|_A \leq D$ for some $D > 0$. For each $K \subseteq \Delta(A)$, let $b_K \in A_c$ such that $b_K|_K \equiv 1$ and I_K be the ideal $\{b \in A : \widehat{b}(K) = \{0\}\}$. Therefore, for each $b \in A$, $bb_K - b_K \in I_K$. Considering the quotient norm of A/I_K , one gets $\|b_K + I_K\|_{A/I_K} = \lim_\alpha \|b_K e_\alpha + I_K\|_{A/I_K} = \lim_\alpha \|e_\alpha + I_K\|_{A/I_K} \leq D$. So, there is some $b \in A_c \cap I_K$ such that $\|b_K + b\|_A < D + \epsilon$. Note that for $a_K := (b_K + b)$, $a_K|_K \equiv 1$ and $a_K \in A_c$.

Conversely, for each $\epsilon > 0$ and $K \subseteq \Delta(A)$ compact, let $a_K \in A$ such that $\widehat{a}_K|_K \equiv 1$ and $\|a_K\|_A \leq D(1 + \epsilon)$. Define $e_{K,\epsilon} := (1 + \epsilon)^{-1}a_K$. It is not hard to show that $(e_{K,\epsilon})_{K,\epsilon}$ forms an approximate identity of the Tauberian algebra A which is $\|\cdot\|_A$ -bounded by D where $\epsilon \rightarrow 0$ and $K \rightarrow \Delta(A)$. \square

Let B be an abstract Segal algebra with respect to a Banach algebra A and A has a $\|\cdot\|_A$ -bounded approximate identity. Then, by an approximation argument, one can show that A has a $\|\cdot\|_A$ -bounded approximate identity which lies in B . If A is a Tauberian algebra, the $\|\cdot\|_A$ -bounded approximate identity of A may belong to $A_c \cap B$. The density condition and relation

of the norms implies that a proper abstract Segal algebra never has a bounded approximate identity.

For a Banach algebra A , let ZA denote the [center](#) of A which is the commutative subalgebra of A consisting of all elements $a \in A$ such that $ab = ba$ for every $b \in A$. The following proposition proves the non-approximate amenability of Segal algebras with notable centres.

Proposition 3 *Let B be a proper abstract Segal algebra of a Banach algebra A which has a central bounded approximate identity and ZB is dense in ZA . If ZA is a regular Tauberian algebra, then B is not approximately amenable.*

Proof. To prove that such an abstract Segal algebra is not approximately amenable, we apply the criterion developed in [2]. To do so, we should construct a sequence $(a_n)_{n \in \mathbb{N}}$ in B such that is $\|\cdot\|_A$ -bounded, $\|\cdot\|_B$ -unbounded, and satisfying $a_n a_{n+1} = a_{n+1} a_n = a_n$ for every $n \in \mathbb{N}$.

First note that ZB is an abstract Segal algebra of ZA . For a fixed $\epsilon > 0$ and $K_0 \subseteq \Delta(ZA)$, for each compact set K such that $K_0 \subseteq K \subseteq \Delta(ZA)$, there is some $a_K \in ZA_c$ such that $\widehat{a_K}|_K \equiv 1$ and $\|a_K\|_A \leq D + \epsilon$, by Lemma 2. Define the $\|\cdot\|_A$ -bounded net $(a_K)_{K_0 \subseteq K \subseteq \Delta(ZA)}$ as above directed by inclusion over compact sets K . Therefore $a_{K_1} a_{K_2} = a_{K_1}$ if $\text{supp}(\widehat{a_{K_1}}) \subseteq K_2$.

We claim that $(a_K)_{K_0 \subseteq K \subseteq \Delta(ZA)}$ is $\|\cdot\|_B$ -unbounded. Note that ZB is a Tauberian algebra. Therefore, A has a $\|\cdot\|_A$ -bounded approximate identity $(e_\alpha) \subseteq ZA_c \cap ZB$. So, for each α , for $K = \text{supp}(e_\alpha)$, $e_\alpha a_K = e_\alpha$. Hence,

$$\|e_\alpha\|_B = \lim_{K_0 \subseteq K \rightarrow \Delta(ZA)} \|e_\alpha a_K\|_B \leq \limsup_{K_0 \subseteq K} \|e_\alpha\|_A \|a_K\|_B.$$

Therefore, if $(a_K)_{K_0 \subseteq K}$ is $\|\cdot\|_B$ -bounded, $(e_\alpha)_\alpha$ is a $\|\cdot\|_B$ -bounded approximate identity of A which violates the properness of B .

To generate a sequence which satisfies the desired conditions mentioned before, fix a non-empty compact set $K_0 \subseteq \Delta(ZA)$. By our claim, we inductively construct a sequence of compact sets $K_0 \subset K_1 \subset \dots$ in $\Delta(ZA)$ such that $a_{K_n} a_{K_{n-1}} = a_{K_{n-1}}$ and $\|B\| a_{K_n} \geq n$ for all $n \in \mathbb{N}$. Then B is not approximately amenable. \square

Now we can prove the main theorem of the paper.

Proof of Theorem 1. Note that for every SIN group G , $L^1(G)$ has a central bounded approximate identity. Moreover, for each Segal algebra $S^1(G)$, $ZS^1(G)$ is dense in $ZL^1(G)$, [7, Theorem 2]. On the other hand, [8, Theorem 1.8] implies that $ZL^1(G)$ is a semisimple regular commutative Tauberian algebra. So Proposition 3 can be applied to finish the proof. \square

Question. Applying some results about structure of locally compact groups, Kotzmann, [7], showed that for every Segal algebra $S^1(G)$, $ZS^1(G)$ is dense in $ZL^1(G)$. The group structure in his proof is essential. It seems that there is not any immediate argument to generalize this proof for a wider class of abstract Segal algebras. It would be of interest if one can generalize

this result to abstract Segal algebras. In other words, is there an abstract Segal algebra whose centre is not dense in its ancestor?

Question. Is every proper Segal algebra of a locally compact group not approximately amenable?

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